

Modulation instability and solitons in two-color nematic crystals

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Abstract

The conditions under which stable evolution of two nonlinear interacting waves are derived within the context of nematic crystals. Two cases are considered: plane waves and solitons. In the first case, the modulation instability analysis reveals that while the nonlocal term suppresses the growth rates, substantially, the coupled system exhibits significantly higher growth rates than its scalar counterpart. In the soliton case, the necessary conditions are derived that lead the solitons to exhibit stable, undistorted evolution, suppressing any breathing behavior and radiation, leading to soliton mutual guiding.

Keywords:

two-color nematicons, nonlocal systems, modulation instability, soliton pairs, soliton guiding

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1. Introduction

Nonlinear wave propagation in optical media is normally the result of a certain balance between dispersion/diffraction and nonlinearity [1]. This balance can lead to special structure formations which are termed solitons or solitary waves. While these structures are proven to be very stable, the necessary balance needed to form them is not always achieved. As a result, due to the dominance of dispersive/diffractive spreading or nonlinear focusing an unwanted growth of initial waves can be observed. This amplitude modulation is widely known and studied in the context of nonlinear Schrödinger equation (NLS) systems [2, 3].

The NLS system, however, for several physically relevant contexts turns out to be an oversimplified description as it cannot model, for example, gain and loss which are inevitable in any physical system [4, 5]. Hence, in order to model important classes of physical systems in a relevant way, it is necessary to go beyond the standard NLS description. To this end it is important to consider adding higher-order terms into the NLS equation, so as to accurately describe waves in dissipative nonlinear systems. In fact, for specific classes of these higher-order terms the resulting system may still be integrable [6] and one can utilize the mathematical tools also provided for the simple NLS case.

In some physical systems simply adding higher order and/or dissipative terms will not result in a better description of their properties. These are, for example, systems with different nonlinear response such as nonlocal. Nonlocality refers to the reorientational response

of the medium to the pulses having a width much greater than the width of the light beams. One of the most interesting materials with nonlocal nonlinear response are nematic liquid crystals. In this context, nonlocality is highly tuneable and solitons, termed nematicons, have been observed induced by a regular local nonlinearity, a nonlocal nonlinearity or a combination of both [7]. Furthermore, nonlinear nonlocal equations similar to those governing nematicons have been found to govern solitary waves in other media, for instance, colloidal [8], thermal [9], photorefractive [10] and plasmas [11].

One of the fundamental effects of wave propagation in nonlinear media is modulation instability (MI). MI was first described in the context of water waves and is referred to as the Benjamin-Feir instability [12]. This is the instability of nonlinear plane waves against small perturbations resulting in their exponential growth. It has been observed experimentally and identified mathematically in various physical applications (other than water waves) [13].

Much like soliton (or coherent structure) formation, MI exists due to the interplay between nonlinearity and dispersion/diffraction. Thus, it highly depends on the nature of the nonlinearity, e.g. cubic, quartic, saturable, nonlocal, etc [14, 15]. Furthermore, the effect is significantly enhanced in coupled systems [16]. We intend to examine the effects of MI in nonlocal media and in particular in two-color nematicons. In these media (in the scalar case), the nonlocal term seems to suppress the growth rate and the overall effect of MI [17, 9]. In addition, exact stationary (soliton) solutions of the coupled equation may still exist under specific requirements. These are termed paired solitons since they specify the intensity profiles of both beams and occur in pairs [18, 19, 20, 5].

The article is divided in sections as follows: In Section 2 we define the equations that govern propagation in these media and perform the MI analysis. Interestingly, we find that the stability criteria are similar to those of the relative NLS system and do not depend on the nonlocal parameter. We also identify two critical values: the maximum growth rate and the critical wavenumber which defines the range of wavenumber that can induce the instability. By doing so, we also identify a critical value for the nonlocality which corresponds to the minimum value needed to stabilize a continuous wave of specific wavenumber. In Section 3, the soliton solutions are discussed and appropriate conditions that relate the relative amplitudes, in order for undistorted evolution to occur, are derived. Finally, we summarize our findings in Section 4.

2. Modulation instability analysis

The governing equations that describe two polarised, coherent light beams of two different wavelengths propagating through a cell filled with a nematic liquid crystal read, in non-dimensional form [21, 22],

$$i\frac{\partial u}{\partial z} + \frac{d_1}{2}\frac{\partial^2 u}{\partial x^2} + 2g_1\theta u = 0 \quad (1)$$

$$i\frac{\partial v}{\partial z} + \frac{d_2}{2}\frac{\partial^2 v}{\partial x^2} + 2g_2\theta v = 0 \quad (2)$$

$$\nu\frac{\partial^2 \theta}{\partial x^2} - 2q\theta = -2(g_1|u|^2 + g_2|v|^2) \quad (3)$$

The variables u and v are the complex valued, slowly varying envelopes of the optical electric fields and θ is the optically induced deviation of the director angle. Diffraction is represented by d_1, d_2 and nonlinearity by g_1, g_2 . Importantly, these variables are allowed to vary their signs while all other constants are taken positive. When the signs of diffraction and nonlinearity are opposite, i.e. $d_1 g_1, d_2 g_2 < 0$, the system is termed defocusing and focusing otherwise. The location of the relative signs is not important; multiplying Eqs. (1)–(2) by -1 and changing $z \rightarrow -z$ moves the sign difference from one place to the other. The nonlocality ν measures the strength of the response of the nematic in space, with a highly nonlocal response corresponding to ν large. The parameter q is related to the square of the applied static field which pre-tilts the nematic dielectric [23, 24, 25]. Note, that the above system corresponds to the nonlocal regime with ν large, where the optically induced rotation θ is small [25]. We will be using dimensionless parameters here because the system above may also be used to describe light propagation in nonlocal media, in general. For nematic crystals, in particular, d_1, g_1, d_2, g_2, q are $O(1)$ while ν is $O(10^2)$ [22, 26].

In order to investigate the MI of a pair of coupled waves we first consider the continuous wave (cw) solution of Eqs. (1)–(3), i.e.

$$u = u_0 e^{2ig_1\theta_0 z}, \quad v = v_0 e^{2ig_2\theta_0 z}, \quad \theta_0 = \frac{g_1 u_0^2 + g_2 v_0^2}{q}$$

where u_0 and v_0 are real constants. Now consider a small perturbation to this cw solution

$$u(z, x) = [u_0 + u_1(z, x)]e^{2ig_1\theta_0 z}, \quad v(z, x) = [v_0 + v_1(z, x)]e^{2ig_2\theta_0 z} \quad (4)$$

which we insert into the system (1)–(3). In order to simplify the analysis we first solve Eqs. (1)–(2) for θ and substitute in Eq. (3). While this eliminates one dependent variable it raises the overall order of the system and as such it only proves useful in the MI analysis where plane wave solutions are investigated (as solutions of a linear system). When solitons or exact solutions are the object of the analysis this is not recommended. The linearized equations (where terms of order u_1^2 and v_1^2 or higher have been neglected) for the small perturbing terms are found to be

$$4iq \frac{\partial u_1}{\partial z} - 2i\nu \frac{\partial^3 u_1}{\partial z \partial x^2} - d_1 \nu \frac{\partial^4 u_1}{\partial x^4} + 2d_1 q \frac{\partial^2 u_1}{\partial x^2} + 8g_1^2 u_0^2 (u_1 + u_1^*) + 8g_1 g_2 u_0 v_0 (v_1 + v_1^*) = 0 \quad (5)$$

$$4iq \frac{\partial v_1}{\partial z} - 2i\nu \frac{\partial^3 v_1}{\partial z \partial x^2} - d_2 \nu \frac{\partial^4 v_1}{\partial x^4} + 2d_2 q \frac{\partial^2 v_1}{\partial x^2} + 8g_2^2 v_0^2 (v_1 + v_1^*) + 8g_1 g_2 u_0 v_0 (u_1 + u_1^*) = 0 \quad (6)$$

Notice that the terms involving ν , that induce the nonlocality, are higher order derivatives; without these terms the problem simply reduces to the linearized NLS problem. This means that their contribution is expected to be highly nontrivial, as they produce higher order polynomials in the dispersion relation; the effect of these terms will become more prominent below. Eqs. (5)–(6) admit solutions of the form

$$u_1(z, x) = c_1 e^{i(kx - \omega z)} + c_2 e^{-i(kx - \omega z)}, \quad v_1(z, x) = c_3 e^{i(kx - \omega z)} + c_4 e^{-i(kx - \omega z)} \quad (7)$$

provided the dispersion relationship

$$p_1(k)\omega^4 + p_2(k)\omega^2 + p_3(k) = 0 \quad (8)$$

with

$$\begin{aligned} p_1(k) &= 16(k^2\nu + 2q) \\ p_2(k) &= -4\nu(d_1^2 + d_2^2)k^6 - 8q(d_1^2 + d_2^2)k^4 + 64(d_1g_1^2u_0^2 + d_2g_2^2v_0^2)k^2 \\ p_3(k) &= d_1^2d_2^2\nu k^{10} + 2d_1^2d_2^2qk^8 - 16d_1d_2(d_2g_1^2u_0^2 + d_1g_2^2v_0^2)k^6 \end{aligned}$$

Eq. (8) is a bi-quadratic and can be solved analytically to produce $\omega = \omega(k)$ as

$$\omega = \pm \sqrt{\frac{-p_2(k) \pm \sqrt{p_2^2(k) - 4p_1(k)p_3(k)}}{2p_1(k)}} \quad (9)$$

MI will be exhibited when this ω has complex solutions: their imaginary part, $\text{Im}\{\omega\}$, will give the relative growth rate, as suggested by Eq. (7). To classify the nature of ω (real or complex) we need to solve a system of inequalities to ensure Eq. (8) only admits real solutions, thus avoiding any exponential growth. In particular, there are three polynomials in k which one needs to prove are positive [27]. This will provide the appropriate conditions for stability. These are:

$$\begin{aligned} \Delta(k) &= 65536k^{14}(2q + \nu k^2)[\nu^2(d_1^2 - d_2^2)^2k^8 + 4q\nu(d_1^2 - d_2^2)^2k^6 \\ &\quad + 4(d_1^2 - d_2^2)(d_1^2q^2 - 8d_1g_1^2\nu u_0^2 + d_2(-d_2q^2 + 8g_2^2v_0^2\nu)k^4 \\ &\quad - 64q(d_1^2 - d_2^2)(d_1g_1^2u_0^2 - d_2g_2^2v_0^2)k^2 + 256(d_1g_1^2u_0^2 + d_2g_2^2v_0^2)^2]Q_1(k) \\ P(k) &= 128(2q + \nu k^2)Q_2(k) \\ D(k) &= 64(2q + \nu k^2)[-4\nu(d_1^2 + d_2^2)k^6 - 8q(d_1^2 + d_2^2)k^4 + 64(d_1g_1^2u_0^2 + d_2g_2^2v_0^2)k^2]^2Q_3(k) \end{aligned}$$

Hence it is sufficient to show that the polynomials

$$\begin{aligned} Q_1(k) &= (d_1^2d_2^2\nu)k^4 + (2d_1^2d_2^2q)k^2 - 16d_1d_2(d_2g_1^2u_0^2 + d_1g_2^2v_0^2) \\ Q_2(k) &= \nu(d_1^2 + d_2^2)k^4 + 2q(d_1^2 + d_2^2)k^2 - 16(d_1g_1^2u_0^2 + d_2g_2^2v_0^2) \\ Q_3(k) &= \nu^2(d_1^2 - d_2^2)^2k^8 + 4q\nu(d_1^2 - d_2^2)^2k^6 \\ &\quad + 4(d_1^2 - d_2^2)[d_1^2q^2 - 8d_1g_1^2u_0^2\nu + d_2(-d_2q^2 + 8\nu g_2^2v_0^2)]k^4 \\ &\quad - 64(d_1^2 - d_2^2)q(d_1g_1^2u_0^2 - d_2g_2^2v_0^2)k^2 + 256(d_1g_1^2u_0^2 - d_2g_2^2v_0^2)^2 \end{aligned}$$

are always positive. This is obtained through Sturm's theorem [28]; since the coefficients of the highest order terms are positive and of even degree, it is sufficient to demand that these polynomials do not exhibit real roots. This happens when:

$$(i) \frac{g_1^2u_0^2}{d_1} + \frac{g_2^2v_0^2}{d_2} < 0 \quad \text{and} \quad (ii) \quad d_1g_1^2u_0^2 + d_2g_2^2v_0^2 < 0 \quad (10)$$

It is now trivial to show that for both conditions to hold it is sufficient to pose that both d_1 and d_2 are negative. As such, the coupled system also follows the condition of stability of the single equation and stability is achieved iff the system is fully defocusing.

In Fig. 1 we depict the unstable evolution of a perturbed unit amplitude wave, as Eqs. (4). In order to avoid falling into the scalar case ($u = v$) we take $d_1 = 1/2$, $\nu = 10$ and all other parameters equal to unity. The computations follow the ETDRK4 method of Ref. [29] as appropriate for stiff problems, since stability issues are discussed. For the plane wave evolution the computational domain $x \in [-4\pi, 4\pi]$, $z \in [0, 20]$ is used and the initial conditions are described in Fig. 1.

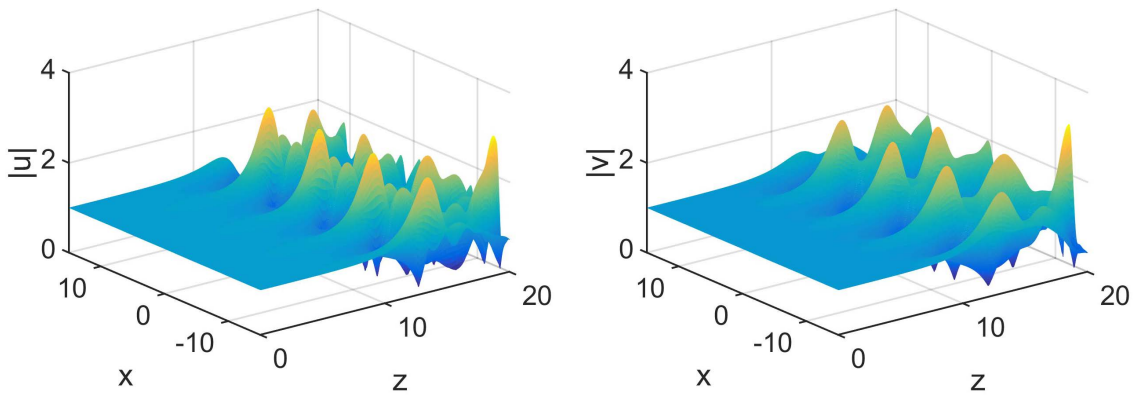


Figure 1: (Color Online) Unstable evolution under Eqs. (1)–(3). Here $u_1(x) = v_1(x) = 0.001e^{i\pi x/4}$, $d_1 = 1/2$, $\nu = 10$ and all other of Eq. (4) parameters are equal to unity.

As clearly seen, the small initial perturbation, u_1, v_1 , results to the exponential growth of the constant amplitude background, u_0, v_0 ; for short times and before the nonlinear terms become important. When ν is increased the constant background will keep its shape for longer times as also suggested by Eq. (9).

We remain on the focusing case where the system is unstable and study the relative growth rates. As mentioned above, the nonlocal term involving ν seems to have a stabilizing effect in the sense that growth rates, in the scalar case, are significantly smaller and MI will need more propagating distance to occur. This feature is preserved here as well, as seen in Fig. 2. In all figures, the growth rate is defined as $\text{Im}\{\omega\}$, while ω is obtained from Eq. (9). We return to this shortly.

The nonlocal term ν has a profound effect on the dynamics of plane waves. While the system is still unstable for large values of ν the range of wavenumbers that cause instability is significantly narrower and in addition the maximum growth rate is smaller. That means that for nematic crystals in particular, where $\nu = O(10^2)$, MI may be suppressed by increasing the value of the nonlocality or by choosing wavenumbers outside this narrow band that result in unstable propagation.

However, the coupling provides significantly higher growth rates as seen in Fig. 3 than the scalar system. As also expected, following this observation, the pure NLS system ($\nu = 0$) has the higher growth rates, so much so that even the single equation surpasses the coupled

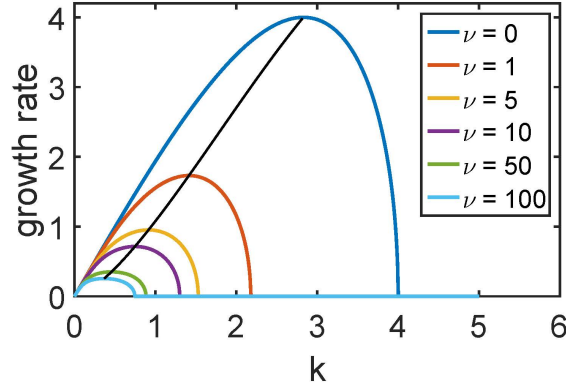


Figure 2: (Color Online) The growth rate, $\text{Im}\{\omega\}$ of Eq. (9), of the focusing system for different values of the nonlocality factor ν . The black solid line corresponds to the maximum growth rate. All other parameters are kept equal to unity.

nonlocal system – cf. Fig. 3 (right). Furthermore, and contrary to the NLS, it has been shown, for the scalar case, that nonlocality of arbitrary shape can indeed eliminate collapse in all physical dimensions [30].

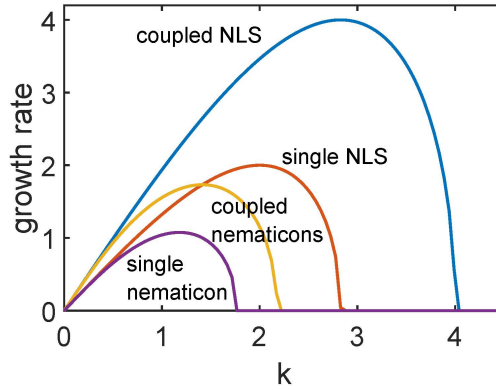


Figure 3: (Color Online) Comparison of the growth rates to the relative NLS equations (Eqs. (1)–(3) with $\nu = 0$). All other parameters are kept equal to unity.

2.1. The relative NLS system

The stability criteria found for Eqs. (1)–(3) are independent of the nonlocal parameter ν . As such, it is relative to investigate the NLS system that corresponds to the case $\nu = 0$, i.e.

$$i\frac{\partial u}{\partial z} + \frac{d_1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{2}{q}(g_1^2|u|^2 + g_1g_2|v|^2)u = 0 \quad (11)$$

$$i\frac{\partial v}{\partial z} + \frac{d_2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{2}{q}(g_1g_2|u|^2 + g_2^2|v|^2)v = 0 \quad (12)$$

Following the same steps in the analysis above (or simply setting $\nu = 0$ everywhere) we obtain the dispersion relation

$$32q\omega^4 - 8k^2[(d_1^2 + d_2^2)qk^2 - 8(d_1g_1^2u_0^2 + d_2g_2^2v_0^2)]\omega^2 + d_1d_2k^6[2d_1d_2qk^2 - 16(d_2g_1^2u_0^2 + d_1g_2^2v_0^2)] = 0$$

which can also be shown (albeit in a much simpler manner than before) to have real solutions iff

$$(i) \frac{g_1^2u_0^2}{d_1} + \frac{g_2^2v_0^2}{d_2} < 0 \quad \text{and} \quad (ii) \quad d_1g_1^2u_0^2 + d_2g_2^2v_0^2 < 0$$

i.e. when d_1 and d_2 are both negative. This is a major difference between this system and the coupled NLS equations (with arbitrary coefficients) [16]. The NLS system is richer in dynamics, as stability may also be obtained even when one of the equations is focusing, or in contrast, the system may be unstable in the purely defocusing regime. In fact, systems much like Eqs. (11)–(12) that are symmetric in the nonlinear coefficients (and $d_1 = d_2$) have been studied extensively in the context of nonlinear optics and birefringent optical fibers, in particular [4]. Nonetheless, this confirms our findings that indeed the NLS ($\nu = 0$) is included in the stability criteria above.

The case $d_1 = d_2$ simplifies the analysis above significantly as the signs of the relative polynomials are easier to find. The interesting observation is that while the above results still hold in this case, the numeric value of $d_1 = d_2$ becomes irrelevant and only its sign is important. Also, important is to notice that since these results are independent of the parameter ν , meaning that the coupled nematicon system and the coupled NLS –with the appropriate coefficients– share the same stability conditions. The nonlocality does not affect the stability criterion, it only contributes to the growth rate, $\text{Im}\{\omega\}$, and its effect is to slow down the occurring instability as shown in Fig. 3.

2.2. Critical wavenumbers and nonlocality values

In the MI analysis above, one can identify some critical numbers that play a key role in the understanding of these results. First and foremost, we identify the so-called maximum growth rate. This value corresponds to the maxima of Fig. 2 and can be found by differentiating Eq. (9), solving the equation $\omega'(k) = 0$ for $k = k_{\text{max}}$ and substituting back to $\omega_{\text{max}} = \omega(k_{\text{max}})$. The change of $\text{Im}\{\omega_{\text{max}}\}$ with ν is shown both in Figs. 2 and 4. However, there is another value that may be interpreted in two ways. We define a critical wavenumber, k_c , which is essentially the greatest wavenumber for which instabilities can occur. To find this critical value one needs to solve the inequality below for k :

$$\frac{-p_2(k) \pm \sqrt{p_2^2(k) - 4p_1(k)p_3(k)}}{2p_1(k)} < 0.$$

Then the critical value can be identified as the solution of

$$d_1d_2\nu k^4 + 2d_1d_2qk^2 - 16(d_2g_1^2u_0^2 + d_1g_2^2v_0^2) = 0. \quad (13)$$

In Fig. 4, we show the dependence of these critical values with the nonlocality ν .

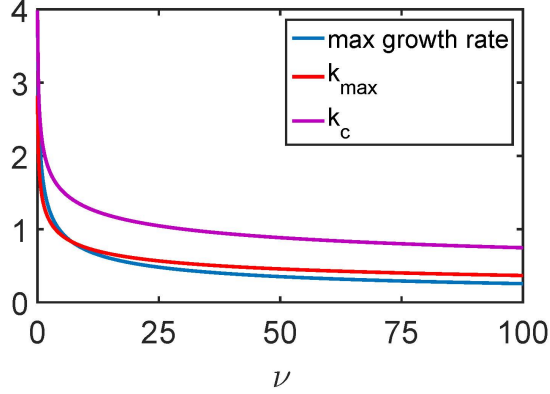


Figure 4: (Color Online) Critical wavenumbers and growth rates and their dependence on the nonlocality.

In particular, for the values of the figures above we find that

$$k_c = \sqrt{\frac{\sqrt{32\nu + 1}}{\nu} - \frac{1}{\nu}}$$

which also demonstrates the way this value is affected by the nonlocality. The relative k_{\max} (and from that ω_{\max}) value can be obtained from the solution of the equation $\nu^2 k^6 + 4\nu k^4 + 4k^2 - 32 = 0$.

Finally, and following the analysis of Ref. [31] one can seek the critical value of the nonlocality parameter that stabilizes a cw of particular wavenumber. This is retrieved again from Eq. (13) only now we solve for ν , i.e.,

$$\nu = 2 \frac{-d_1 d_2 q k^2 + 8(d_2 g_1^2 u_0^2 + d_1 g_2^2 v_0^2)}{d_1 d_2 k^4}$$

For example, for the values as above, the wave number $k = 1$ will always correspond to a stable cw iff $\nu \geq 30$, as also confirmed by Figs. 2 and 4. This critical value is not a general criterion for stability as it depends on the particular wavenumber. This means that while the particular critical value of ν may stabilize a specific wavenumber another value will render the system unstable, consistently with the analysis above. Only when the system is full defocusing MI is absent.

3. Solitons

The dynamics of two-color nematicons propagation and interactions in the nonlocal limit is usually studied using variational method based on an appropriate trial function/ansatz whose parameters (amplitude, width, etc) are chosen so that the Lagrangian of the system is minimized [26, 32, 22]. However, since these are not exact solutions they are expected to shed diffractive radiation much like the solutions of the regular NLS system. We intend to

remedy this by finding exact solutions to Eqs. (1)–(3) and the conditions associated with these solutions.

As seen above the coupled system is, in the focusing case, unstable. This means that any initial condition is subject to instability. Thus, it is natural to seek conditions under which soliton solutions may exist that will not undergo the instability process. To find these solutions (if they exist), we assume that the stationary solutions of the system (1)–(3) take the form

$$u(z, x) = a_1 \operatorname{sech}^2(bx) e^{i\mu_1 z}, \quad v(z, x) = a_2 \operatorname{sech}^2(bx) e^{i\mu_2 z}, \quad \theta(x) = a_3 \operatorname{sech}^2(bx).$$

Substituting directly into Eqs. (1)–(3), we obtain the expressions for the soliton parameters:

$$\mu_1 = \frac{qd_1}{\nu}, \quad \mu_2 = \frac{qd_2}{\nu}, \quad b = \sqrt{\frac{q}{2\nu}}, \quad a_3 = \frac{3q\lambda}{4\nu}. \quad (14)$$

The solitons' amplitudes are related through

$$g_1 a_1^2 + g_2 a_2^2 = \lambda \frac{9q^2}{8\nu} \quad (15)$$

subject to the condition

$$\frac{d_1}{g_1} = \frac{d_2}{g_2} = \lambda. \quad (16)$$

Although the freedom of one free parameter (which also relates amplitude and velocity in the NLS case) is not redeemed, another property is obtained. We are now able to control the amplitude of one of the components through Eq. (15). However, this is again a fundamental difference with the NLS system. Indeed, in the NLS equations it is straightforward to obtain a variety of different cases and soliton types (bright and/or dark). Here, only the focusing case can produce bright solitons while we were not able to find dark solitons in this manner [33]. They may, however, be obtained, in the small amplitude limit, using the methods of Ref. [34]. One final comment is that this procedure may also be used for more than two equations relating the relative amplitudes through an equation of the form of Eq. (15).

The role of the nonlocal term ν is profound here as well. In particular, when $\nu \gg 1$, as is the case for liquid crystals, it may become (experimentally) more difficult to obtain soliton solutions and radiation free propagation. Indeed, from Eq. (15) the smaller the right hand side becomes the more difficult it becomes to obtain amplitudes for soliton propagation. In fact, when this term vanishes it results in $a_1 = a_2 = 0$, i.e. no soliton solutions exist. Furthermore, Eq. (15) suggests that

$$a_1^2 \leq \lambda \frac{9q^2}{8\nu g_1}, \quad a_2^2 \leq \lambda \frac{9q^2}{8\nu g_2},$$

meaning that even with the freedom to choose one of the amplitudes that cannot exceed this maximum value. With $\nu \gg 1$ it is further implied that solitons can only exist in the small amplitude limit, thus suggesting that for these systems the analysis of Ref. [34] may be more appropriate.

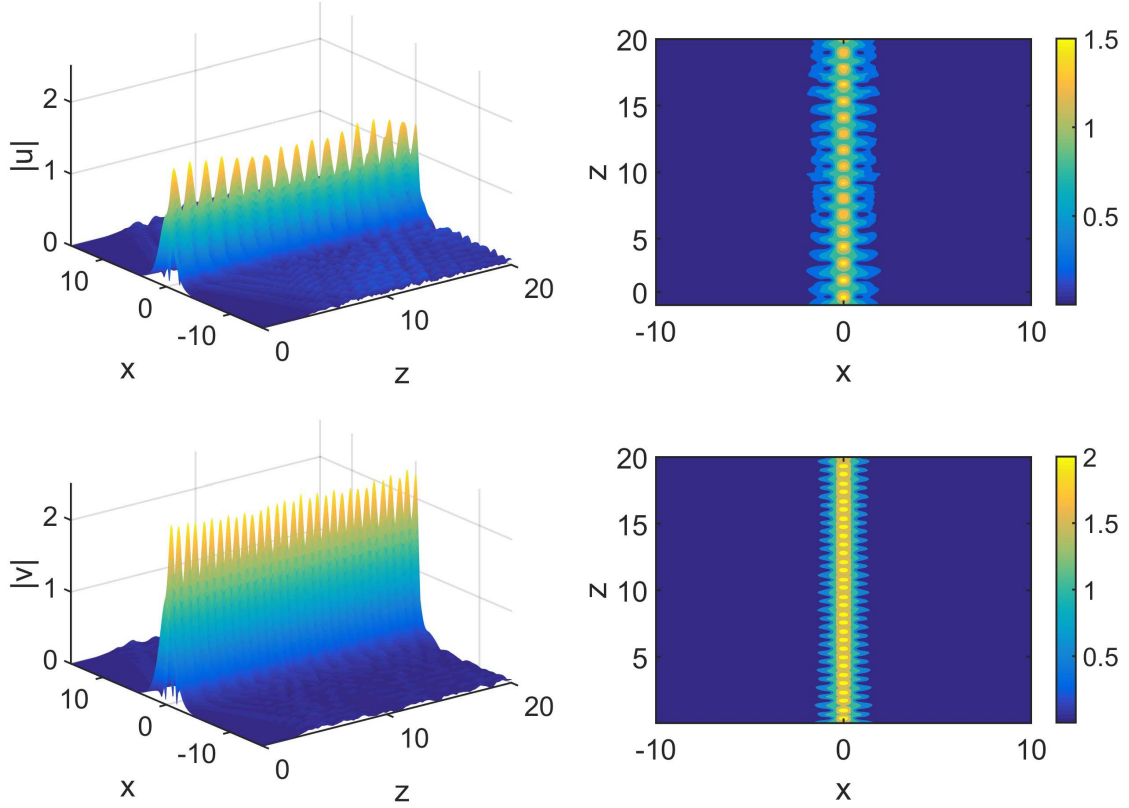


Figure 5: (Color Online) A soliton evolution undergoing a breathing behavior. Here we used $d_1 = g_1 = 1$, $d_2 = g_2 = 2$, $\nu = 1$, $q = 1$, $a_1 = 1$ and $a_2 = 1.5$.

To illustrate the difference we evolve a pulse which does not obey the amplitude relation Eq. (15) in Fig. 5.

Notice here that the initial condition almost immediately starts a breathing behavior and shedding of radiation. On the other hand if we choose initial conditions that obey the amplitude condition, Eq. (15), the result is a stable typical solitonic evolution as shown in Fig. 6.

To numerically test the robustness of these structures we repeat the above calculation with the initial conditions of Fig. 6 with 20% random noise added. The resulting propagation is depicted in Fig. 7. Notice, that despite the initial noise the pulses maintain their structural integrity and propagate without radiating. This is strong evidence that these structure are stable; the complete stability analysis will be provided in a later communication.

Note, that there is a qualitative difference between the evolutions depicted in Figs. 5 and 7. In the first, where a pulse that does not obey the amplitude condition, undergoes a breathing behavior but it is not dispersed into radiation because of the nonlocal nature of the system. In the second, the soliton is exhibiting breathing, but at the amplitude of the noise making the propagation neutrally stable. This feature can be used to distinguish pulses in the experimental realization, as noise is an unavoidable part of any experiment.

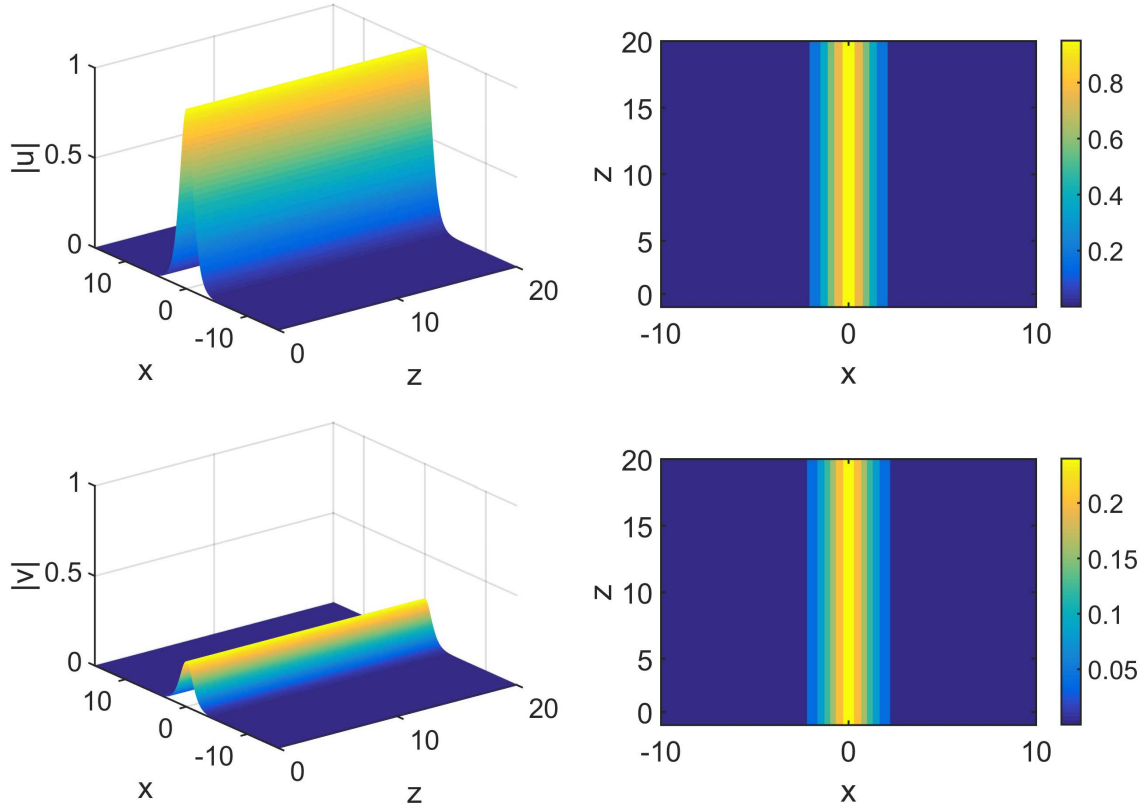


Figure 6: (Color Online) A typical soliton evolution. Parameters are same as in Fig. 5 only now a_2 is obtained from Eq. (15).

4. Conclusions

To conclude, we have derived the appropriate conditions for the modulation instability of plane waves propagating in nonlocal media. It is found that the stability properties follow the ones of the single system while growth rates are significantly higher. In addition, much like the scalar case, the nonlocality has a profound effect in the suppression of the effect as it results in significantly smaller growth rates. Useful information have also been extracted from the study of the critical values of the analysis: the maximum growth rate and the critical wavenumber which defines the range of wavenumbers that can induce the instability. In addition, we defined another critical value based on the nonlocality parameter, which defines the range of values that can stabilize a cw solution of a particular wavenumber.

In the soliton case, we find that bright solitons may exist strictly on the focusing case and only when a particular condition holds for the relative amplitudes. Any other pulse will undergo a breathing behavior but will not disperse into radiation due to the nonlocal nature of the equations. This condition allows for soliton mutual guiding and may help in their experimental realization as the correct soliton perturbed with noise will exhibit breathing, but at the amplitude of the noise making the propagation neutrally stable.

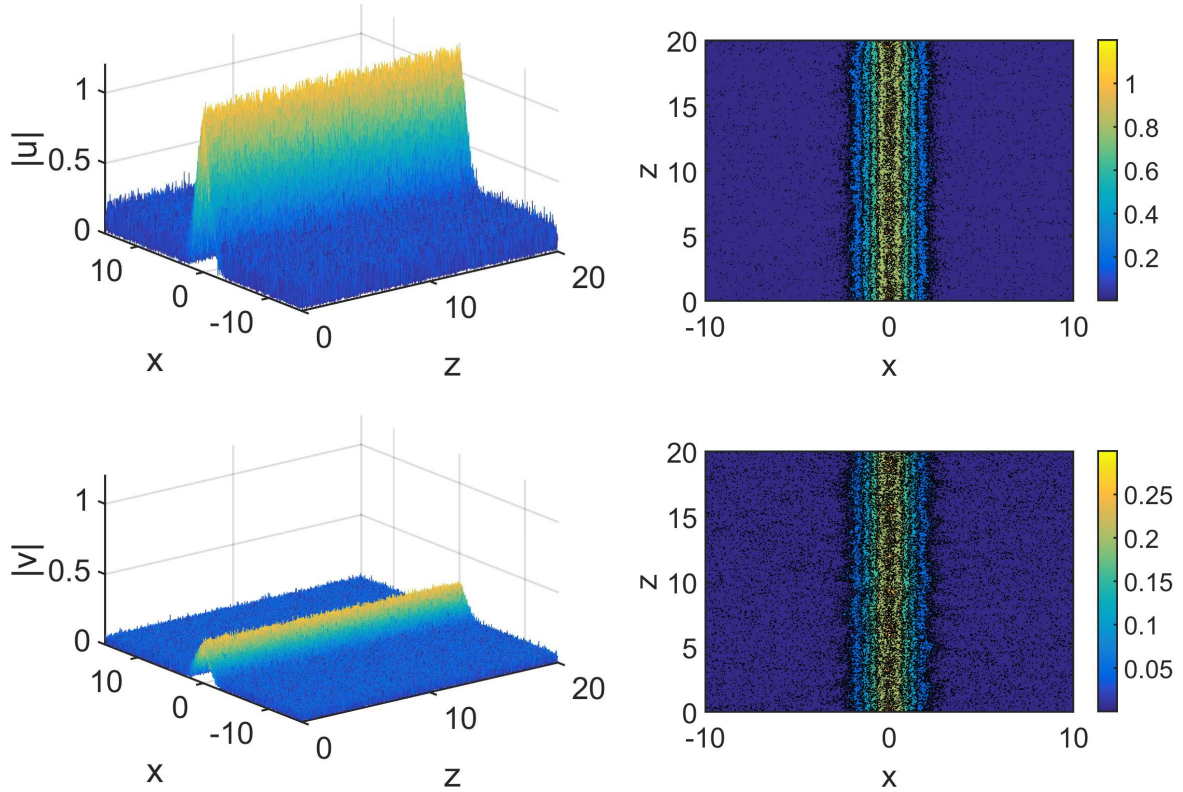


Figure 7: (Color Online) The soliton evolution of Fig. 6 with 20% random noise added, to numerically test the stability of these solutions.

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